MODEL OF THE FRACTAL PROFILE OF A FATIGUE CRACK

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At present, fatigue fracture (FF) is described by means of empirically established laws (such as the Paris law [1, 2]). The lack of universal physical models of the FF process is making it more difficult to properly set up experiments to explain the effect of the microsocpic and mesoscopic characteristics of the material on FF. In the present investigation, we attempt to describe FF as the diffusion limited aggregation (DLA) [3, 4] of microdamages in the plastic region at the crack tip. This allows us to construct the model of fatigue-crack propagation on the basis of iterations of a discrete random mapping. This approach to the description of FF requires the use of characteristics – such as the fractal dimension (FD) of the FF surfaces (characterizing their geometric similitude [5]) – which differ from those traditionally used in fracture mechanics.

1. Locality of the FF Process. The fatigue fracture of macroscopic specimens (parts etc.) occurs at stresses much lower than the yield point σ_y due to the singular behavior of the stresses near the crack tip. This behavior conforms to the Irwin solutions in the elastic region [1]. The boundary of the plastic zone around a crack tip is usually determined from the von Mises criterion

$$3J_2 = \sigma_y^2 \tag{1.1}$$

(J_2 is the second invariant of the stress deviator). In the proposed model, it is assumed that all FF processes occur inside a plastic region with a boundary determined by Eq. (1.1).

In reviewing the large amount of fractographic data for fatigue fracture surfaces [2], it can be concluded that the propagation of fatigue cracks is not a continuous process but is more likely a discrete, incremental event. This conclusion is based specifically on the fact that the number of cycles per characteristic fractographic dimension in fatigue tests is much greater than unity. In other words, the crack advances after a certain degree of damage to the material has been attained in the plastic zone during cyclic loading. The life of the plastic zone can be estimated from the data in [6], for example. Here, it was shown that, under the same experimental conditions, the dependence of fatigue-crack velocity on the number of cycles correlates with the corresponding relations for certain acoustic-emission characteristics. In this case, there is a lag of 5000 cycles (the correlation coefficient is greater than 0.94). Meanwhile, the product of this value and crack velocity roughly corresponds to the size of the plastic zone.

2. Fatigue Fracture as a DLA of Microdamages. The accumulation of damages is one of the channels for dissipating the mechanical energy supplied to the specimen. We will assume that this process has an affinity with DLA and we will use the well known DLA model in [3] based on evolutionary equations for concentrations of clusters of microdamages C_M and the number M of associated individual damages (such as vacancies):

$$\frac{dC_M}{dt} = \frac{1}{2}\sum_{j=1}^{M} K(j,\ M-j)\,C_j C_{M-j} - C_M \sum_{i=1}^{\infty} K(M,i)\,C_i. \tag{2.1}$$

It should be noted that the DLA is a rather general theory, in view of the fact that the range of the forces joining the "particles" into an aggregate is much smaller than the size of the aggregate (not only due to the nature of these forces in the case of short-range action, but also due to the shielding effect). Thus, the properties of the aggregate should be independent of the nature of the binding forces [7]. The cluster–cluster DLA model in [3] is analogous to a description of processes involving nucleation and annihilation, while

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the formulation of the problem on the basis of equations for rates of change in concentration implies that the unions of microdamages are kinetic in character.

The model being proposed here for the fracture of the plastic zone in front of the tip of a fatigue crack can be described as follows. Microdamages accumulate and combine in the plastic zone, the evolution of these microdamages being described by Eqs. (2.1) with the multiplicative kernel $K(i, j) \sim (i, j)^{\omega}$. At $\omega > 1/2$, a single cluster "survives" from among the numerous clusters of different dimensions and attaches itself to other clusters [4]. Intuitively, this case corresponds to ductile FF. The solutions of Eqs. (2.1) have the properties:

$$\langle M \rangle \sim t^z,\ C_M \sim M^{-\tau} t^{-w},\ z = 1/(1 - 2\omega),\ (2 - \tau)z = w \tag{2.2}$$

(the angle brackets denote statistical averaging).

Also, considering the expediency of assuming that both the damage to the material [8] and aggregates [4] is of a fractal nature, we put

$$M \sim (R_M)^D \tag{2.3}$$

(R_M is the size of an aggregate M, while D represents the fractal dimension FD). We use properties (2.2) to find $(R_{\langle M \rangle})^D \sim \langle M \rangle \sim t^z$. Considering that the crack advances when $R_{\langle M \rangle}$ reaches a critical value proportional to crack length ℓ (as is the size of the plastic zone — see below) we obtain

$$dl/dN \sim R_{\langle M \rangle}/t \sim (R_{\langle M \rangle})^{1-D/z} \sim l^{1-D/z}$$

(N is the number of cycles). Comparison with the Paris law $d\ell/dN \sim K_I^n \sim \ell^{n/2}$ (K_I is the stress intensity factor (SIF) [1]) gives

$$n = 2 + D2(2\omega - 1). \tag{2.4}$$

An expression has also been found for the dependence of the density of total strain energy in FF tests ΔW_f on the number of cycles to fracture N_f [9]:

$$\Delta W_f = kN_f^{\alpha} + c \tag{2.5}$$

(k, c, and α are material constants). This expression can be explained as follows. Let us suppose that a crack moves in uniform increments s over a certain crack-length interval from ℓ_0 to ℓ_c, where fracture occurs. We further assume that the strain-energy density for each increment is W_1. In an approximation of cylindrical symmetry with the symmetry axis along the crack front, we have $W_1 \sim s^2$ for the case of loading of the crack by the first mode. Then the total strain-energy density will be the sum of the crack-growth increments. At a cyclic-loading frequency ν, with allowance for (2.2), (2.3) we obtain

$$\sum^{N_s} W_1 \sim \sum^{N_s} s^2 \sim \sum^{N_s} t_1^{2z/D} \sim \sum^{N_s} (N_1 \nu)^{2z/D} \sim \sum^{N_s} (N_f \nu / N_s)^{2z/D}$$

(N_s is the number of increments, while t_1 and N_1 are the time and number of cycles with elapse for each increment). This simple model gives the relationship between the FD and the exponent α:

$$(2\omega - 1)D = -2/\alpha. \tag{2.6}$$

According to the data in [9], $\alpha = -0.665$. From this, for $\omega = (1 - 2/(\alpha D))/2$ we have $\omega \approx 1.5 > 1/2$. The latter result qualitatively reflects the above assumption regarding ω and the ductile character of fracture of the low-alloy steel for which Eq. (2.5) was obtained [9] (steel ASTM A 516 Gr 70). Thus, the proposed model provides a qualitative interpretation of the negativity of the exponent α.

Taken together, Eqs. (2.4) and (2.6) can be regarded as a "one-power" scaling analogous to the two-power scaling in phase transformation theory. This implies that FF is regarded as one of the critical phenomena for which the formation of fractal structures is a natural event under critical conditions. The above-described FF model also makes it possible to adopt the simplification that the crack advances over equal time intervals by the amount

$$s \sim l^{\nu},\ \gamma \cong n/2. \tag{2.7}$$

This assumption can be used to model the macroscopic dynamics of FF. The case $\omega < 1/2$, intuitively corresponding to brittle fracture, necessarily makes the model of crack-tip advance considerably more complex.

3. Model of Fatigue-Crack Propagation. For the sake of definiteness, we will examine a fatigue crack loaded by the first mode. As is known [1], fatigue cracks are nucleated in planes that are not perpendicular to the direction of the maximum tensile stresses. However, the macroscopic crack propagates perpendicularly to these stresses. Thus, in order to use the von Mises criterion (1.1) to isolate a local FF region, it is necessary to write the stresses in the form of the superposition of at least two loading modes [1, 10]:

$$\begin{gathered}
\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - A\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\right],\\
\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + A\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right],\\
\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + A\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right],\\
\sigma_z = \nu(\sigma_x+\sigma_y),\ A = K_{II}/K_I
\end{gathered} \tag{3.1}$$

(all of the notation is standard). Then the second invariant of the deviator has the form

$$J_2 = \frac{\nu^2-\nu+1}{3}(\sigma_x+\sigma_y)^2 - \sigma_x\sigma_y + \tau_{xy}^2. \tag{3.2}$$

Substitution of Eqs. (3.1), (3.2) into (1.1) gives us the equation of the boundary of the local FF zone $r(\theta)$. Within this zone, elastic solutions (3.1) – as the criteria of linear fracture mechanics – are invalid. However, under conditions of self-similar crack growth corresponding to the second stage of FF – when crack growth is controlled by K_I – it can be assumed that the course of FF processes occurring in the plastic zone is determined by its size and configuration. We will assume that damage accumulation in the zone occurs as DLA described by Eqs. (2.1) with the multiplicative kernel ($\omega > 1/2$). Here, the damage cluster breaks up in the direction of the maximum of $r(\theta)$ and the crack-growth increment obeys (2.7). Then the advance of the crack can be modeled by iterations of a discrete mapping:

$$\begin{gathered}
\varphi_{m+1} = \varphi_m + \theta_m^M,\quad l_{m+1} = l_m + s_m\cos\varphi_m,\ K_{I,m+1} = \sigma_\infty\sqrt{\pi l_{m+1}},\\
K_{II}/K_I = \lambda(l)\sin(-\varphi_m),\ m = 1, 2, \ldots .
\end{gathered} \tag{3.3}$$

Here, φ_m is the angle between a normal to the tensile stresses at the distance from the crack tip σ_∞ and the deflected crack tip for the increment m; θ_m^M is the angle corresponding to the maximum $r(\theta)$; $\lambda(\ell)$ is a function accounting for the relationship between the first and second loading modes with a change in φ_m and crack length. We will assume that K_I is independent of φ_m [11], while $K_{II} \cong \sigma_\infty \sin(-\varphi_m)\sqrt{\pi s}$. For example, it can be assumed that $\lambda(\ell) = \chi\sqrt{s/\ell}$, where χ accounts for the intensification of the second mode at inclusions, grain boundaries, and other discontinuities.

Unfortunately, if we attempt to simplify (1.1) after the insertion of (3.1), (3.2) so as to be able to find θ^M analytically, we find that we can do this only for an incompressible medium ($\nu = 1/2$). In this case,

$$J_2 = (\sigma_x - \sigma_y)^2/4 + \tau_{xy}^2.$$

The same expression for J_2 is obtained if it is assumed that the stress tensor and deviator have four components. In this case, there are two maxima differing by π in the function $r(\theta)$ describing the boundary of the plastic zone. Of these two, we choose the one corresponding to advance of the crack in a direction which compensates for the existing deviation from the main plane of the crack. Such a choice is justified by the fact that – as can easily be proven – the material between the deflected crack tip and the main direction of the crack (perpendicular to σ_∞) undergoes tension in two directions x and y, while the material on the

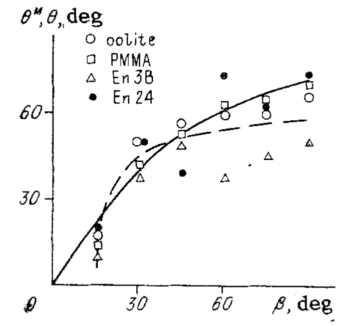

Fig. 1

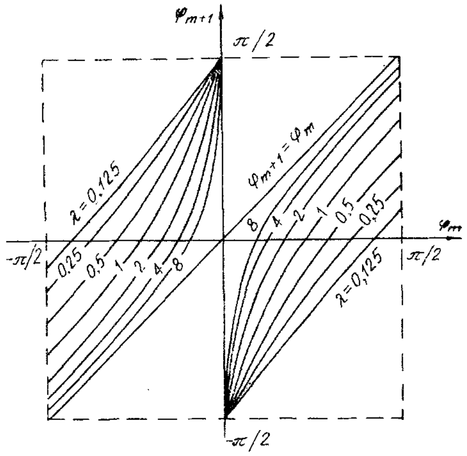

Fig. 2

other side of the tip undergoes compression in one direction and tension in another. With allowance for this, at $\nu = 1/2$, we find

$$r(\theta) = \frac{3K_{\mathrm{I}}^2}{2\pi\sigma_{\mathrm{T}}^2}\left[\frac{\sin^2\theta}{4}(1 + A^2) + A\cos\theta(\sin\theta + A\cos\theta)\right],$$
$$\theta_m^M = \mathrm{arctg}\left(\mathrm{sign}(-\varphi_m)\sqrt{z^2+1} - z\right),\; z = (3A^2 - 1)/(4A),$$
$$A = K_{\mathrm{II}}/K_{\mathrm{I}} = \lambda(l)\sin(-\varphi_m). \tag{3.4}$$

Figure 1 shows data from [11] for the four-point bending of beams of different materials, with $\beta = \pi/2 - \varphi$. The solid line shows the relation for the angle of crack advance θ corresponding to the Erdogan–Sih criteria [11], while the dashed line corresponds to (3.4) with the ratio K_{II}/K_I calculated for the conditions of the experiment in [11]. The satisfactory agreement – particularly at small β (large φ) – validates the use of (3.4). To complete the construction of the model, we need to assign the value of s. This quantity will be calculated in accordance with (2.7) from the formula $s = \xi(r(\theta^M))^\gamma$. We will assume that the crack advances by the maximum amount $r_c(\theta^M)$ at $K_I = K_{Ifc}$ or

$$r_c(\theta^M) = r_c^M = \frac{3K_{\mathrm{I}fc}^2}{2\pi\sigma_{\mathrm{T}}^2}.$$

It follows from this that

$$s = r_c^M\left[r(\theta^M)/r_c^M\right]^\gamma \tag{3.5}$$

(r_c^M can be calculated in terms of the critical crack length ℓ_c corresponding to $K_{Ifc} = \sigma_\infty\sqrt{\pi\ell_c}$).

Having recorded the position of the crack after 100 crack increments with $(\sigma_\infty/\sigma_y)^2 = 9.26\cdot 10^{-4}$, $\ell_c = 50$ mm, an initial crack length $\ell_0 = 5$ or 15 mm, and $\gamma = 1$-3, we numerically integrated (3.3)-(3.5) and obtained a linear curve in the coordinates $\log(d\ell/dN) - \log(K_I)$ (the kinetic curve of fatigue fracture [2]). With an accuracy better than 0.2%, the slope of this curve (the Paris index) was determined to be $n = 2\gamma$. This result is indicative of the "good" random properties of the mapping described in detail above.

4. Stochastic Properties of the Mapping $\varphi_{m+1}(\varphi_m)$. To study the properties of this mapping, we will simplify (3.3)-(3.5) by fixing λ. Figure 2 shows the function $\varphi_{m+1}(\varphi_m)$ for different λ. The discontinuity at $\varphi_m = 0$, due to the symmetry of the plastic zone, makes the mapping similar to the so-called Bernoulli shift [12] – a piecewise-linear random mapping the iteration of which is equivalent to a certain realization of a sequence of coin tosses. Figure 3 presents a bifurcation diagram of the mapping $\varphi_{m+1}(\varphi_m)$ with fixed λ. The diagram gives an illustration of its invariant density and is nonuniform even at values $\lambda > 0.3$ cor-

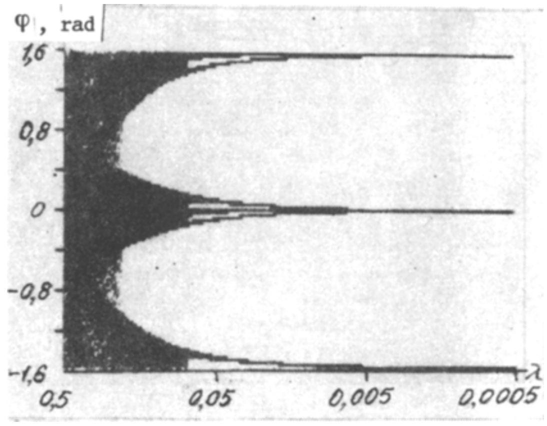

Fig. 3

TABLE 1

λ	D
1,5	1,0881
1,0	1,0635
0,5	1,0605
0,2	1,1560

TABLE 2

γ	D	$\Delta\lambda$
1	1,0951	0,3—0,5
2	1,1202	0,2—0,3
3	1,1524	0,13—0,15

responding to the quote "good" random regimes. However, even at lower λ – when "branches" are clearly visible on the diagram – the Lyapunov index of this mapping remains small but positive and is roughly equal to λ, i.e., the mapping $\varphi_{m+1}(\varphi_m)$ is random. This by itself justifies the selection criterion θ^M and opens up the possibility of statistically accounting for the effect of given features of the material on FF by introducing additional parameters into the model.

The mapping $\varphi_{m+1}(\varphi_m)$ constructed using the Erdogan–Sih criterion (the direction of the maximum σ_θ [11]) yields only one bifurcation at $\lambda = 1$ and is stable at $\lambda < 1$ when $\varphi_m = 0$. This makes it possible to use this criterion to construct a universal FF model. As in the case of the general form of J_2 (3.2), it is not possible to analytically find the angle θ^T ($\varphi_{m+1} = \varphi_m + \theta^T$) by means of the Tresca criterion (nucleation of cracks in accordance with the angles of the maximum τ_{xy}).

Thus, the actual process of fatigue-crack propagation is modeled as the steady state of a certain dynamic system corresponding [13] to steady state of mapping (3.3)-(3.5). The study of this mapping is an independent problem (such as due to the spontaneous disturbance of symmetry at $0.01 < \lambda < 0.1$ in Fig. 3, which might be connected with the nonconservative nature of the mapping, i.e., the absence of the Hamiltonian; this qualitatively reflects the dissipative character of fatigue fracture) and lies outside the scope of the present study.

5. Fractal Properties of the Model. The randomness of mapping (3.3)-(3.5) makes it possible to suggest that geometric similitude exists between model crack profiles, i.e., that they are fractal in nature. It can be seen from Fig. 2 that for large λ the "trajectory" of a crack in the plane $(\varphi_m, \varphi_{m+1})$ will remain in the first or third quadrants longer than when λ is smaller, i.e., λ should have an effect on the geometric characteristics of the crack profile (such as its FD).

Table 1 shows data from the calculation of the FD of profiles by the method described in the appendix for different fixed λ and $\gamma = 1.5$. It is apparent that there is some correlation between λ and the fractal dimension D. At $\lambda < 0.2$, the "self-similar" shallow section of the fractal graph acquires the character of a discontinuity. This is a manifestation of the quasiperiodic properties of the mapping – wanderings among the "branches."

In the case of the complete mapping (3.3)-(3.5) and fixed χ, the FD of the profiles depends on γ. Table 2 shows data on the correlation of γ and D at $\chi = 20$ [$\Delta\lambda$ is a sample range of the function $\lambda(\ell)$]. The positive correlation between γ and D agrees qualitatively

TABLE 3

Method		Measured D	Deviation
Minkowsky—Buligand	[16]	1,356	0,144
Cell counting		1,413	0,087
Spectral-density method		1,449	0,051
Method of horizontal structural elements		1,537	0,037
Variation method		1,495	0,005
Our method		1,4698	0,0302

with Eq. (2.4), since it can be reliably assumed that $n = 2\gamma$ (as was already noted). With the advance of the crack, λ increases as $\ell^{(\gamma-1)/2}$. Thus, in principle, it is possible to model the transition from the periodic regime of fatigue-crack propagation (fatigue striae [2]) to completely random regimes with $\lambda > 0.3$. The proposed model also makes it easy to fix the size of the plastic zone to reflect structural features of certain materials.

The proposed approach to modeling the propagation of fatigue cracks, based on discrete random mappings, offers a wide range of possibilities for incorporating empirical FF laws (the Paris law, fatigue striae, etc.) [14] into the model. There are no serious obstacles to generalizing the model to the case of complex stress fields or the simultaneous propagation of several fatigue cracks. This in turn opens up the possibility of numerically studying the fatigue fracture of specific parts and structures.

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Appendix. Direct Calculation of Fractal Profiles. Many algorithms have now been developed to calculate fractal dimensions, the most accurate being algorithms which employ the expansion method [15, 16]. However, the latter require a considerable amount of machine time because they in essence involve the analysis of an irregular two-dimensional object. This contrasts with algorithms which employ the sheaf method [15] and deal directly with the profile in numerical form. One shortcoming of the sheaf method is the irregularity of the fractal graph and, thus, the relatively low accuracy of calculation of the FD. Described below is a method which can be considered intermediate between the above two types of algorithms, since it involves the analysis of a "unidimensional" profile but is similar to algorithms based on the expansion method with regard to the logic of its construction.

In actual studies, researchers deal with profiles that have been transformed after digitization into a sequence of segments which intersect at points with specified coordinates — segments of a broken line. One variant of the expansion method entails calculation of the number of disks of a certain size placed along the line in such a way as to touch both one another and the line. Here, the FD is found from the slope of the double-log dependence of the number of disks on their size [5, 15]. Having moved the chain of disks in such a way as to center the greatest possible number of disks on the line and having joined their centers by other segments, we obtain a second broken line which has segments of a specified length and approximates the first broken line. The same approximation can be obtained in a simpler and more illustrative manner: having chosen a point on the initial broken line, lay off a certain-size segment from this point such that the other end of the segment belongs to the initial broken line and a piece of the profile of the greatest possible length is enclosed between its ends; lay off the next segment from this end, and so forth. This chain approximates the initial broken line in a manner similar to the chain obtained after connecting the centers of the displaced disks. However, the approach just described is distinctly different from the sheaf method, as can be seen from the figures in [5, 15, 17].

The algorithm for constructing such an approximation of the initial broken line (profile) is elementary and does not require the same amount of machine time as the expansion-method algorithm [16]. In addition, the algorithm quite reliably reproduces empirical measurements of profile roughness on a profilometer with different-size sensitive elements [18]. The FD is determined from the slope of the dependence of $\log(N\varepsilon/L')$ on $\log\varepsilon$ (N is the number of segments of length ε required to approximate a broken line with a projection of the length

L' [18]). Table 3 shows data from calculation of the FD corresponding to the Quiswetter curve with the use of different methods. Also shown are the deviations from the exact value D = 1.5. The proposed method gives a regular fractal graph with a correlation coefficient of 0.9881 for the points on the straight section [14]. It can be seen from Table 3 that it is the equal of expansion algorithms with respect to accuracy in determining the FD.

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